Mixing and chemical reaction in the laminar wake of a flat plate

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(*Received* 6 *June* 1957)

SUMMARY

The initial value problem presented by mixing and chemical reaction in the wake of a flat plate is solved using the boundary-layer approximation. When a cool combustible mixture and its hot combustion products are separated by a finite, perfectly insulating flat plate, the velocity, temperature, and combustible concentration are determined in the vicinity of the trailing edge.

The mixing problem without chemical reaction is solved in terms of a 'universal solution' for a given initial temperature ratio and Prandtl number from which the solution for arbitrary temperature ratios can be obtained.

The mixing problem with chemical reaction is solved in terms of a 'universal solution' for the first two terms of an assumed series solution for the temperature. In this case the 'universality' is with respect to a parameter B characterizing the chemical and hydrodynamic initial conditions.

The axial distance from the trailing edge to the first local temperature maximum is given in terms of the initial conditions and is shown to be greatly shortened by the presence of the viscous wake as compared with non-viscous mixing.

1. INTRODUCTION

When flames are stabilized on bluff bodies, the interaction of a cool combustible mixture with its hot combustion products is of great importance for the stabilization mechanism (Zukoski & Marble 1955; Cheng & Kovitz 1958). An idealization of the problem was first carried out by Marble & Adamson (1954). They consider a non-viscous, perfectly insulating, semi-infinite partition separating a cool combustible mixture from its hot combustion products. At the trailing edge of the partition velocity, temperature and concentration of combustible are uniform in each half-plane. Temperature distribution and combustible concentration are, initially (i.e. at the trailing edge), step functions while the velocity distribution may or may not be a step function. Their analysis then gives the temperature, combustible and velocity distributions in the neighbourhood of the trailing edge of the semi-infinite partition. Of particular interest is the distance downstream from the trailing edge at which a local temperature maximum, due to heat release by chemical reaction, first appears.

The present analysis is concerned with the identical problem except for one important difference. The partition between the two streams is now taken to be of finite length and viscosity is included. As a result, the initial velocity distribution is no longer uniform but of the Blasius type. It will be seen that this significantly alters the distribution of stream properties in the immediate neighbourhood of the trailing edge as compared with Marble & Adamson's result.

2. ASSUMPTIONS CONCERNING THE MODEL AND ANALYTICAL FORMULATION OF THE PROBLEM

In this paper a detailed discussion of the assumptions on which the model is based will be omitted. They have been fully discussed by Adamson (1954). In essence, the equations to be presented for solution describe the following flow, which is schematically shown in figure 1. In the upper half-plane there is, initially, a cool combustible with temperature T_I, density ρ_I, and free stream velocity u_I. The lower half-plane consists, initially, of fluid which is chemically inert with respect to the upper stream (say its combustion products). Its temperature is $T_{II} > T_I$, its free stream velocity is $u_{II} = u_I$, and its density is $\rho_{II} = \rho_I T_I/T_{II}$ since the pressure is assumed constant throughout the field. The specific heat C_p and molecular weight M of each component (combustible and combustion product) are equal, with C_p taken as constant. The transport properties, dynamic viscosity μ, coefficient of thermal conductivity λ, and binary diffusion coefficient D of each component are equal and vary as though the molecules were Maxwellian with the Eucken correction for λ holding. From the above it follows that the Prandtl number $Pr = C_p \lambda/\mu$ and Schmidt number

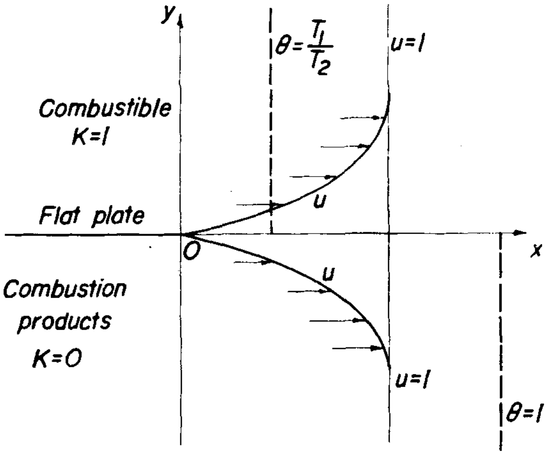

Figure 1. Schema of model with coordinate system.

$Sc = \mu/\rho D$ are constant throughout the field. Also, $\rho\mu$ = constant. Heat release by chemical reaction is assumed to be first order, that is, the heat release Q per unit volume per unit time is assumed to be given by the equation

$$Q = \frac{\Delta H}{\tau}\rho K e^{-A/RT},$$

where ΔH = heat release per unit mass of combustible, ρ = local density, mass per unit volume, K = relative mass concentration of combustible, mass of combustible per unit mass of mixture, τ = characteristic chemical time constant, R = universal gas constant, A = activation energy, a constant. Finally, the flow is assumed steady with a laminar mixing region in which the usual boundary layer approximations hold (Goldstein 1938).

The governing equations are obtained from conservation of mass, conservation of momentum, conservation of energy, and conservation of combustible. Since $\rho\mu$ = constant the Howarth transformation (Howarth 1948) may be used to uncouple the continuity and momentum equations from the energy and combustible equations. The equations to be solved can then be written as

$$\left.\begin{aligned} u_x + v_y &= 0, \\ uu_x + vu_y &= u_{yy}, \end{aligned}\right\} \qquad (1)$$

$$\left.\begin{aligned} u\theta_x + v\theta_y &= \frac{1}{Pr}\theta_{yy} + BKe^{-\theta_a/\theta}, \\ uK_x + vK_y &= \frac{1}{Sc}K_{yy} - CKe^{-\theta_a/\theta} \end{aligned}\right\} \qquad (2)$$

where $B = 4l\Delta H/u_I \gamma C_p T_{II}$, $C = 4l/u_I\tau$. $\theta = T/T_{II}$, $\theta_a = A/RT_{II}$, and l is the length of the flat plate. The symbols u and v denote non-dimensional axial and transverse velocity components in the incompressible plane while x, y are the corresponding distance coordinates, with origin at the trailing edge of the flat plate. Their definitions are taken from Goldstein (1930).

Equations (1) have been solved by Goldstein (1930) for the case of flow in the wake of a flat plate with symmetric Blasius velocity profiles at the trailing edge. This solution will be used in solving (2).

Goldstein's solution for u, v is expressed in terms of the independent variables

$$\xi = x^{1/3}, \qquad \eta = y/3\xi.$$

These variables will also be used in the solution for θ and K. Furthermore, since the Blasius initial condition is in terms of the Blasius series $u_0(y)$, valid only for $y > 0$, Goldstein's solution for u, v is valid only for $\eta > 0$. It is therefore necessary to construct solutions for θ and K in the two half-planes separately and join them smoothly at $y = 0$. (Quantities in the upper half-plane will be denoted by an overbar.)

Another consequence of the Blasius series initial condition is that Goldstein's solution is expressed in two forms, one valid for small y, the other valid for large y. Therefore, solutions for θ and K must also be constructed in the same way.

The boundary conditions for θ and K are

$$\begin{aligned} \lim{}_1 \bar{\theta}(\xi,\bar{\eta}) &= \lim{}_2 \bar{\theta}(\xi,\bar{\eta}) = T_I/T_{II} = \Lambda, \\ \lim{}_3 \theta(\xi,\eta) &= \lim{}_4 \theta(\xi,\eta) = 1, \\ \lim{}_1 \bar{K}(\xi,\bar{\eta}) &= \lim{}_2 \bar{K}(\xi,\bar{\eta}) = 1, \\ \lim{}_3 K(\xi,\eta) &= \lim{}_4 K(\xi,\eta) = 0, \end{aligned}$$

where

$$\begin{aligned} \lim{}_1 &\equiv \xi \to 0, \quad \bar{\eta} \to \infty, \quad \bar{y} = \text{const.}, \\ \lim{}_2 &\equiv \bar{\eta} \to \infty, \quad \bar{y} \to \infty, \\ \lim{}_3 &\equiv \xi \to 0, \quad \eta \to \infty, \quad y = \text{const.}, \\ \lim{}_4 &\equiv \eta \to \infty, \quad y \to \infty. \end{aligned}$$

3. SOLUTION FOR θ AND K FOR SMALL y

For small y, solutions of the form

$$\left.\begin{aligned} \theta(\xi,\eta) &= \sum_{n=0}^{\infty} \theta_n(\eta)\xi^n, \\ K(\xi,\eta) &= \sum_{n=0}^{\infty} K_n(\eta)\xi^n, \end{aligned}\right\} \tag{3}$$

are assumed, with similar forms in the upper half-plane.

Since the initial condition as $\xi \to 0$ with $y =$ constant is independent of y it can be shown that the first term of each series must be independent of ξ and all terms with $n \geqslant 1$ must vanish as $\xi \to 0$, $y =$ constant. In terms of the assumed solutions (3) this implies

$$\left.\begin{aligned} \lim_{\bar{\eta}\to\infty} \bar{\theta}_0 &= \Lambda, & \lim_{\eta\to\infty} \theta_0 &= 1, \\ \lim_{\bar{\eta}\to\infty} \bar{K}_0 &= 1, & \lim_{\eta\to\infty} K_0 &= 0, \end{aligned}\right\} \tag{4}$$

and

$$\lim_{\bar{\eta}\to\infty} \bar{\theta}_n/\bar{\eta}^n = \lim_{\eta\to\infty} \theta_n/\eta^n = \lim_{\bar{\eta}\to\infty} \bar{K}_n/\bar{\eta}^n = \lim_{\eta\to\infty} K_n/\eta^n = 0 \tag{5}$$

for $n \geqslant 1$.

For large y the form of solution must be such that at some intermediate value of y the two solutions join smoothly. Its form will be discussed after the solution for small y has been obtained.

Goldstein (1930) gives, for small y,

$$\left.\begin{aligned} u &= \tfrac{1}{3} \sum_{l=0}^{\infty} b_{3l+1} \xi^{3l+1}, \\ v &= \tfrac{1}{3} \sum_{k=0}^{\infty} c_{3k-1} \xi^{3k-1}, \end{aligned}\right\} \tag{6}$$

where

$$b_{3l+1} = f'_{3l}(\eta), \qquad c_{3k-1} = \eta f'_{3k}(\eta) - (3k+2) f_{3k}(\eta).$$

The functions f_0, f_3, f_6 and their derivatives were calculated by Goldstein and have been recalculated for smaller mesh size in the present work. Substituting (3) and (6) into (2), and equating the coefficients of like powers of ξ to zero in the usual manner, we obtain a series of equations for $\theta_n(\eta)$ and $K_n(\eta)$:

$$\left.\begin{aligned} \theta_s'' + \lambda_2 f_0 \theta_s' - \lambda_s f_0' \theta_s &= PrI_s, \\ K_s'' + \gamma_2 f_0 K_s' - \gamma_s f_0' K_s &= ScH_s, \end{aligned}\right\} \tag{7}$$

where

$$\begin{aligned} \lambda_s &= sPr, \quad \gamma_s = sSc, \qquad (s = 0, 1, 2...) \\ I_0 &= I_1 = H_0 = H_1 = 0, \\ I_2 &= -9BK_0 e^{-\theta_a/\theta_0}, \\ H_2 &= 9CK_0 e^{-\theta_a/\theta_0}, \end{aligned}$$

and for $s \geqslant 3$

$$I_s = -9B \sum_{\substack{n,j=0 \\ n+j=s-2}} D_j K_n + \sum_{\substack{t=3 \\ l=1 \\ t+3(l-1)=s}} \{(t-3)f_{3l}'\theta_{t-3} - (3l+2)f_{3l}\theta_{t-3}'\}$$

with H_s identically the same except for replacement of B by $-C$, θ_{t-3} by K_{t-3} and θ_{t-3}' by K_{t-3}'. The coefficients D_j are obtained from the expansion

$$e^{-\theta_a/\theta} = \sum_{j=0}^{\infty} D_j(\eta)\xi^j = e^{-\theta_a/\theta_0} + \frac{\theta_a \theta_1}{\theta_0^2} e^{-\theta_a/\theta_0}\xi + \dots .$$

Similar results hold for the upper half-plane.

By the method of variation of parameters a formal solution of (7) may be written as

$$\begin{aligned} \theta_s(\eta) = a_s f_1^{(s)}(\eta) + b_s f_2^{(s)}(\eta) - Pr \int^{\eta} I_s(\eta) \exp\left\{\lambda_s \int^{\delta} f_0(z)\, dz\right\} \times \\ \times \{f_2^{(s)}(\eta) f_1^{(s)}(\delta) - f_1^{(s)}(\eta) f_2^{(s)}(\delta)\}\, d\delta, \end{aligned} \tag{8}$$

where a_s, b_s are arbitrary constants and $f_1^{(s)}$, $f_2^{(s)}$ are linearly independent solutions of the reduced (homogeneous) equation. A similar result holds for $K_s(\eta)$.

For small and large η, f_0 behaves as

$$\left.\begin{aligned} f_0 &= \beta_0 \eta + \beta_0^2 \frac{\eta^3}{3\,!} - 2\beta_0^3 \frac{\eta^5}{5\,!} + \dots, \\ f_0 &\sim \tfrac{1}{2}\alpha_1(\eta + \delta_0)^2 + O[(\eta + \delta_0)^{-4} \exp\{-\tfrac{1}{3}\alpha_1(\eta + \delta_0)^3\}], \end{aligned}\right\} \tag{9}$$

respectively, where $\beta_0 = 3{\cdot}67869$, $\alpha_1 = 5{\cdot}97708$, $\delta_0 = 0{\cdot}3408$ (Goldstein 1930).

For small η a solution of the reduced form of (7) is given by

$$\begin{aligned} \theta_s^{(0)} &= a_s(1 + \gamma_2^{(s)}\eta^2 + \gamma_4^{(s)}\eta^4 + \dots) + b_s(\eta + \gamma_3^{(s)}\eta^3 + \gamma_5^{(s)}\eta^5 + \dots) \\ &= a_s f_1^{(s)} + b_s f_2^{(s)}, \end{aligned} \tag{10}$$

where the $\gamma_i^{(s)}$ are known functions of β_0, λ_2, and λ_s. The meaning of a_s and b_s in (8) is now clear. a_s is the value of θ_s at $\eta = 0$, while b_s is the value of θ_s' at $\eta = 0$.

4. SOLUTION OF THE REDUCED EQUATION FOR LARGE η

The boundary conditions (4) and (5) are all applied as $\eta \to \infty$. Therefore, a_s and b_s cannot be evaluated from the general solution (8) until the asymptotic form of this solution is known. Since $f_1^{(s)}$ and $f_2^{(s)}$ cannot be expressed analytically for all η the asymptotic form of (8) cannot be obtained directly.

Using the asymptotic form of f_0 and neglecting the exponentially small term, we find that the equation (7) for $\theta_s^{(0)}$ becomes

$$\theta_s^{(0)''} + a\sigma^2\theta_s^{(0)'} - sa\sigma\theta_s^{(0)} = 0 \qquad (11)$$

where $a = \alpha_1 Pr$ and $\sigma = \eta + \delta_0$. The solution of (11) gives $\theta_s^{(0)}$ as a function of σ. It can be shown that only exponentially small terms will be added to $\theta_s^{(0)}$ as $\eta \to \infty$ if the exponentially small term in f_0 is retained.

The substitutions

$$\sigma = \left(\frac{3\omega}{a}\right)^{1/3}, \qquad \theta_s^{(0)} = \left(\frac{3\omega}{a}\right)^{-1/3} e^{-\frac{1}{2}\omega} W_s\left(\frac{3\omega}{a}\right)$$

transform (11) into

$$\frac{d^2W_s}{d\omega^2} + \left\{-\frac{1}{4} + \frac{-\frac{1}{3}(s+1)}{\omega} + \frac{\frac{1}{4}-\frac{1}{36}}{\omega^2}\right\} W_s = 0, \qquad (12)$$

which is a form of Whittaker's equation (Whittaker & Watson 1952). In terms of $\theta_s^{(0)}$ and σ the asymptotic form of its solution may be written as

$$\begin{aligned}\theta_s^{(0)} \sim A_s \sigma^{-s-2} e^{-\frac{1}{3}a\sigma^3}\Bigg\{1 &+ \\ &+ \sum_{n=1}^{\infty} \frac{\{\frac{1}{36}-(\frac{1}{3}s+\frac{1}{3}+\frac{1}{2})^2\}\{\frac{1}{36}-(\frac{1}{3}s+\frac{1}{3}+\frac{3}{2})^2\}\ldots\{\frac{1}{36}-(\frac{1}{3}s+\frac{1}{3}+n-\frac{1}{2})^2\}}{n!\,(\frac{1}{3}a\sigma^3)^n}\Bigg\} + \\ &+ B_s\sigma^s\Bigg\{1 + \\ &+ \sum_{n=1}^{\infty} \frac{\{\frac{1}{36}-(-\frac{1}{3}s-\frac{1}{3}+\frac{1}{2})^2\}\{\frac{1}{36}-(-\frac{1}{3}s-\frac{1}{3}+\frac{3}{2})^2\}\ldots\{\frac{1}{36}-(\frac{1}{3}s+\frac{1}{3}-n+\frac{1}{2})^2\}}{n!\,(-\frac{1}{3}a\sigma^3)^n}\Bigg\}\end{aligned} \qquad (13)$$

where A_s and B_s are arbitrary constants. A similar result holds for the upper half-plane and for $K_s^{(0)}$.

5. DETERMINATION OF ARBITRARY CONSTANTS FROM BOUNDARY CONDITIONS

Although a particular solution to the asymptotic form of (7) has not, as yet, been obtained it can be seen from (13) that A_s will not be fixed by the boundary condition for the lower half-plane alone. However, B_s must be determined since the second term of (13) becomes of order σ^s as $\sigma \to \infty$. It will be seen that as $\sigma \to \infty$ the particular solution of (7) becomes exponentially small in the lower half-plane and can be neglected in the upper half-plane due to the assumption of zero heat release by chemical reaction in the cool combustible. This, therefore, implies that for $s \geqslant 1$

$$B_s = \overline{B}_s = 0.$$

Six constants remain to be determined. They are a_s, b_s, A_s, $\bar{a}_s$, $\bar{b}_s$, and $\bar{A}_s$. Their determination stems from the matching of value and slope of θ_s and $\bar{\theta}_s$ at two values of η, namely $\eta = \bar{\eta} = 0$ and $\eta = \bar{\eta} = \eta_m$, η_m being some intermediate value of η and $\bar{\eta}$ where both solutions of θ_s and $\bar{\theta}_s$ are assumed valid.

The matching of value and slope at $\eta = \bar{\eta} = 0$ immediately gives

$$\left.\begin{aligned} a_s &= \bar{a}_s, \\ b_s &= -\bar{b}_s. \end{aligned}\right\} \tag{14}$$

The matching of value and slope at $\eta = \bar{\eta} = \eta_m$ gives

$$\left.\begin{aligned} a_s D_{1s} + b_s D_{2s} + D_{3s} &= A_s D_{4s} + D_{5s}, \\ a_s D'_{1s} + b_s D'_{2s} + D'_{3s} &= A_s D'_{4s} + D'_{5s}, \\ a_s D_{1s} - b_s D_{2s} + D_{6s} &= \bar{A}_s D_{4s} + D_{7s}, \\ a_s D'_{1s} - b_s D'_{2s} + D'_{6s} &= \bar{A}_s D'_{4s} + D'_{7s}, \end{aligned}\right\} \tag{15}$$

where the D_i and D'_i are functions of η evaluated at $\eta = \bar{\eta} = \eta_m$. Thus six equations are available for the determination of the six arbitrary constants.

6. THE FIRST FIVE TERMS OF THE SOLUTION FOR SMALL y

(i) For large η, from equation (13),

$$\theta_0 \sim A_0 \sigma^{-2} \exp(-\tfrac{1}{3}a\sigma^3)\left\{1 + \sum_{n=1}^{\infty} a_n^{(0)} \sigma^{-3n}\right\} + 1, \tag{16}$$

where $a_n^{(0)}$ is a constant given by (13) and $B_0 = 1$ because of the boundary condition (4) and, for large $\bar{\eta}$,

$$\bar{\theta}_0 \sim \bar{A}_0 \sigma^{-2} \exp(-\tfrac{1}{3}a\sigma^3)\left\{1 + \sum_{n=1}^{\infty} a_n^{(0)} \sigma^{-3n}\right\} + \Lambda, \tag{17}$$

since $\bar{B}_0 = \Lambda$.

Carrying out the matching technique described in the previous section, it can be shown that

$$a_0 = \tfrac{1}{2}(1+\Lambda), \quad b_0 = \tfrac{1}{2}(1-\Lambda)/\{D_{20} - D'_{20} D_{40}/D'_{40}\}, \quad A_0 = -\bar{A}_0, \tag{18}$$

where the D symbols are as defined by (15).

Since the D's do not depend upon Λ it can further be shown that

$$\theta_0(\eta, Pr, \Lambda) = \frac{1+\Lambda}{2} + \frac{1-\Lambda}{1-\Lambda^*}\left[\theta_0(\eta, Pr, \Lambda^*) - \frac{1+\Lambda^*}{2}\right] \tag{19}$$

where Λ^* is a particular value of Λ. The solution for K_0 is obtained in the same way with $K_0 = \frac{1}{2}$ when $\eta = 0$.

The function $\theta_0(\eta, Pr, \Lambda^*)$ has been integrated numerically for $Pr = 0{\cdot}91$ and $1{\cdot}00$ with $\Lambda^* = 0{\cdot}286$. Numerical integration is discussed in § 9. Figure 2 shows θ_0 and K_0 for $Sc = 1{\cdot}00$.

Note that

$$K_0 = (1-\theta_0)/(1-\Lambda^*). \tag{20}$$

(ii) In solving for θ_1 through the matching technique it can be shown that a non-trivial solution for a_1 and b_1 will be obtained only if the Wronskian

$f_1^{(1)}f_2^{(1)\prime} - f_1^{(1)\prime}f_2^{(1)}$ vanishes. Since $f_1^{(1)}$ and $f_2^{(1)}$ are linearly independent solutions for θ_1 this implies that the only solution for θ_1 satisfying the boundary conditions is that for which $a_1 = b_1 = 0$ or $\theta_1 \equiv 0$. The same result holds for K_1, namely, $K_1 \equiv 0$.

(iii) The equation for θ_2 is

$$\theta_2'' + 2f_0 Pr\, \theta_2' - 2f_0' Pr\, \theta_2 = -9BPrK_0\, e^{-\theta_a/\theta_0}. \qquad (21)$$

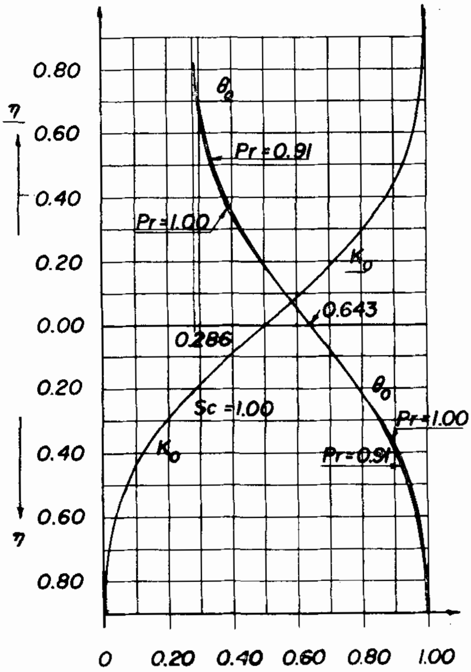

Figure 2. Solution for θ_0 and K_0.

θ_2 is the first factor in the solution for θ which contains the effect of chemical reaction. This is seen from the form of I_2. With chemical reaction absent, $B = 0$. Then $\theta_2 \equiv 0$, by the same argument as was used to show that $\theta_1 \equiv 0$. By direct substitution it can be shown that if $\theta_2(\eta, \ldots, B^*)$ is a satisfactory solution of (21) for $B = B^*$, then, for any B,

$$\theta_2(\eta, Pr, Sc, \theta_a, \Lambda, B) = \frac{B}{B^*}\theta_2(\eta, Pr, Sc, \theta_a, \Lambda, B^*). \qquad (22)$$

Thus the dependence on the parameter B has been extracted. Furthermore, using the matching technique of §5, with $\exp(-\theta_a/\Lambda) \ll 1$, it can be shown that

$$\theta_2 \propto e^{-\theta_a} \qquad (23)$$

for values of η where $\theta_0 \div 1$.

It is interesting to note that if $Sc = Pr$,

$$K_2(\eta, Pr, \theta_a, \Lambda, C) = -\frac{C}{B^*}\theta_2(\eta, Pr, Sc, \theta_a, \Lambda, B^*). \qquad (24)$$

K_2 is, physically, the first term in the concentration solution that gives the effect of loss of combustible due to chemical reaction.

θ_2 has been obtained by numerical integration for $B = 2{\cdot}58 \times 10^{14}$, $\theta_a = 23{\cdot}96$, $Pr = 1{\cdot}00$, and $\Lambda = 0{\cdot}286$. It is shown in figure 3. Outline of the numerical method is deferred to § 9.

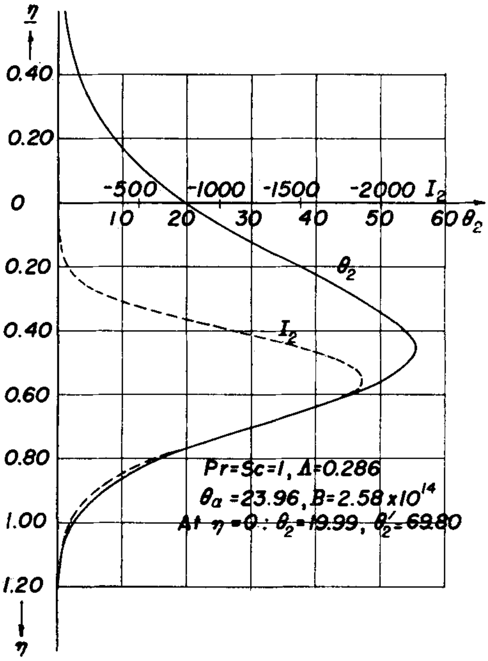

Figure 3. ‘ Temperature distribution ’ θ_2 due to heat release I_2.

From the dependence of I_2 on K_0, it can be seen that I_2 becomes exponentially small as $\eta \to \infty$. Therefore, a particular solution of (21) for large η would also be of exponential order. In the upper half-plane $\theta_0 \to \Lambda$, so that $\bar{I}_2$ is of the order of $B \exp(-\theta_a/\Lambda) \ll 1$, since chemical reaction is assumed to be negligible in the cool combustible. Therefore, a particular solution of (21) for large $\bar{\eta}$ would be of order $B \exp(-\theta_a/\Lambda)$. These observations show that for the boundary conditions to hold, $B_2 = \bar{B}_2 = 0$. Thus, for large η and $a = b$,

$$\theta_2 \sim \left[A_2 \sigma^{-4} \left\{ 1 + \sum_{n=1}^{\infty} a_n^{(2)} \sigma^{-3n} \right\} + \frac{9BPrE_0}{a} e^{-\theta_a} \{ \sigma^{-3} - \tfrac{1}{2} b_1^{(0)} \sigma^{-6} + \ldots \} \right] e^{-\frac{1}{3} a \sigma^3} \tag{25}$$

(iv) Since $\theta_1 = K_1 \equiv 0$, it can be seen from the definition of I_3 that θ_3 is really a term due to mixing without chemical reaction. Furthermore, I_3 and $\bar{I}_3$ are antisymmetric about $\eta = \bar{\eta} = 0$, because θ_0' and $\bar{\theta}_0'$ are antisymmetric about $\eta = \bar{\eta} = 0$. This implies the antisymmetry of θ_3 and $\bar{\theta}_3$ about $\eta = \bar{\eta} = 0$. This conclusion may also be seen from the matching technique of § 5.

Finally, because of (19) and the dependence of I_3 on θ_0',

$$\theta_3(\eta, Pr, \Lambda) = \frac{1-\Lambda}{1-\Lambda^*}\,\theta_3(\eta, Pr, \Lambda^*). \tag{26}$$

θ_3 is exhibited in figure 4 for $Pr = 1{\cdot}00$ and $\Lambda^* = 0{\cdot}286$.

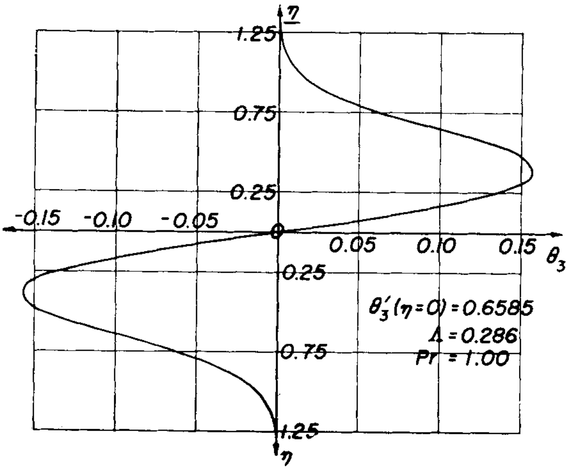

Figure 4. Solution for θ_3.

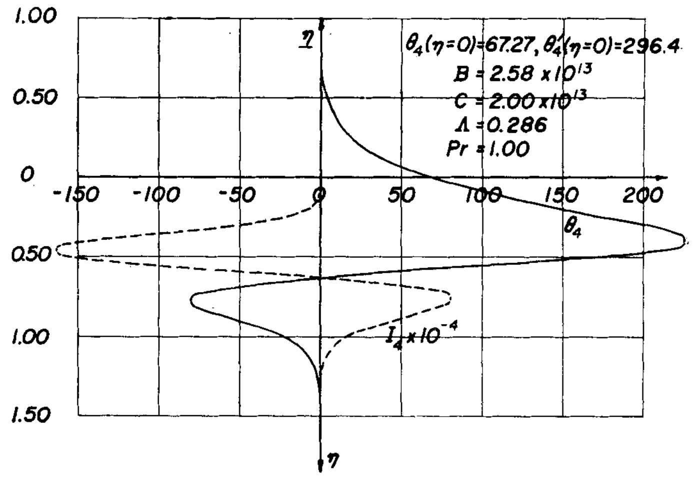

Figure 5. Solution for θ_4, showing I_4.

(v) θ_4 is the last term computed with chemical reaction. A particular solution of the asymptotic form of the governing equation is exponentially small since I_4 becomes exponentially small when $\eta \to \infty$. Therefore, as before, $B_4 = 0$. The same argument as used to show that $\bar{B}_2 = 0$ will show that $\bar{B}_4 = 0$. θ_4 is exhibited in figure 5.

(vi) The solution of $\theta(\xi,\eta)$ for large η and small y will now be collected since it will be useful in the discussion of the next section. If the asymptotic forms of $\theta_s(\eta)$ given by (13), together with the asymptotic form of a particular solution to θ_s, be substituted into (3), the asymptotic form of $\theta(\xi,\eta)$ is obtained. Using θ_0 and θ_2 only, with $Pr = Sc$, i.e. $a = b$, and $\sigma = \eta+\delta_0$,

$$\theta(\xi,\eta) \sim 1+[A_0\{\eta^{-2}-2\delta_0\eta^{-3}+3\delta_0^2\eta^{-4}-\ldots\}+ \\ +\xi^2A_2\{\eta^{-4}-4\delta_0\eta^{-5}+\ldots\}+\ldots]\exp\{-\tfrac{1}{3}a(\eta^3+3\delta_0\eta^2+3\delta_0^2\eta+\delta_0^3)\} \quad (27)$$

after making use of $B_s = \bar{B}_s = 0$. A similar result holds in the upper half-plane and for $K(\xi,\eta)$.

7. SOLUTION FOR LARGE y AND ITS MATCHING WITH SOLUTION FOR SMALL y

It was pointed out in § 2 that Goldstein's (1930) solution for u, v is split into two forms, one valid for small y, the other valid for large y. Accordingly, the solution for θ and K must also be so split and made to match at some intermediate y which is really the outer limit of validity for the small-y solution.

The form of solution for small y at its outer limit of validity is obtained by making η large. This form, for $\theta(\xi,\eta)$, is given by (27). The new solution valid for large y will be forced to match this form when y is reasonably small.

For large y it will be assumed that the terms measuring the rate of heat release due to chemical reaction and the rate of consumption of combustible are small compared with convection, conduction, and diffusion terms. This assumption implies that the region in which Goldstein's wake solution for large y applies is one where pure mixing, without chemical reaction, is predominant. The governing equations (2) are then simplified to

$$u\theta_x+v\theta_y = Pr^{-1}\theta_{yy}, \qquad uK_x+vK_y = Sc^{-1}K_{yy}, \quad (28)$$

with identical equations in the upper half-plane.

Goldstein's velocity solution, for large y, to be inserted here, is

$$\left.\begin{aligned} u &= \psi_0'+\xi\psi_1'+\xi^2\frac{\psi_2'}{2!}+\xi^3\frac{\psi_3'}{3!}+\ldots, \\ v &= -\tfrac{1}{3}\xi^{-2}\psi_1-\tfrac{2}{3}\xi^{-1}\frac{\psi_2}{2!}-\tfrac{3}{3}\xi^0\frac{\psi_3'}{3!}-\ldots, \end{aligned}\right\} \quad (29)$$

where $\quad \psi_0' = u_0(y), \quad \psi_1' = Au_0', \quad \dfrac{\psi_2'}{2!} = \dfrac{A^2}{2!}u_0'', \quad \dfrac{\psi_3}{3!} = \dfrac{A^3}{3!}u_0'''-2yu_0,$

and $A = 3\delta_0$, with $u_0(y)$ representing the Blasius velocity profile at $x = \xi = 0$.

For given sufficiently large η, ξ can be made arbitrarily small. In so doing, y is made arbitrarily small, since $\eta = y/3\xi$. Considering only terms independent of ξ, equation (27) becomes, as $\xi\to 0$,

$$\theta(\xi,\eta) \sim 1+A_0(\eta^{-2}-2\delta_0\eta^{-3}+\ldots)\exp\{-\tfrac{1}{3}a(\eta^3+3\delta_0\eta^2+3\delta_0^2\eta+\delta_0^3)\}. \quad (30)$$

The solution of (28) using the u, v given by (29) must reduce to (30) for $\xi \to 0$ and sufficiently small y. This suggests assuming a solution of the form

$$\theta(\xi, \eta) = 1 + N \sum_{r=0}^{\infty} \Theta_r(y) \xi^{r+2} \exp\{\Theta_a(y)\xi^{-3} + \Theta_b(y)\xi^{-2} + \Theta_c(y)\xi^{-1}\} \tag{31}$$

in the lower half-plane where Θ_a, Θ_b, Θ_c and $\Theta_r(y)$, with $r = 0, 1, 2, \ldots$ are, as yet, undetermined functions of y, and N is an, as yet, arbitrary constant. Substituting this assumed solution and (29) into (28), and equating the coefficients of like powers of $\xi^r \exp\{\Theta_a \xi^{-3} + \ldots\}$ to zero, one gets a series of equations for Θ_r.

The first four equations are

$$\left.\begin{aligned} \frac{1}{Pr}\Theta_a'^2 + \psi_0' \Theta_a &= 0, \\ \frac{2}{Pr}\Theta_b' + \frac{2}{3}\frac{\psi_0'}{\Theta_a'}\Theta_b + \psi_1' \frac{\Theta_a}{\Theta_a'} + \frac{1}{3}\psi_1 &= 0, \\ \frac{2}{Pr}\Theta_c' + \frac{1}{3}\frac{\psi_0'}{\Theta_a'}\Theta_c + \frac{1}{Pr}\frac{\Theta_b'^2}{\Theta_a'} + \frac{2}{3}\psi_1' \frac{\Theta_b}{\Theta_a'} + \frac{\psi_2'}{2!}\frac{\Theta_a}{\Theta_a'} + \frac{1}{3}\psi_1 \frac{\Theta_b'}{\Theta_a'} + \frac{2}{3}\frac{\psi_2}{2!} &= 0, \\ \frac{2}{Pr}(\log\Theta_0)' + \frac{1}{\Theta_a'}\frac{d}{dy}\left[\frac{1}{3}\psi_1\Theta_c + \frac{2}{3}\frac{\psi_2}{2!}\Theta_b - \frac{3}{3}\frac{\psi_3}{3!}\Theta_a - \frac{2}{3}\psi_0\right] + & \\ + \frac{1}{Pr}\left[2\frac{\Theta_b'\Theta_c'}{\Theta_a'} + \frac{\Theta_a''}{\Theta_a'}\right] &= 0. \end{aligned}\right\} \tag{32}$$

Although the equations after the first one remain first order and linear, they rapidly become lengthy.

Solving for Θ_a one gets

$$\Theta_a = -\frac{1}{Pr}\left\{G_1 - \tfrac{1}{2}Pr \int \psi_0'^{1/2}\, dy\right\}^2 \tag{33}$$

where G_1 is an arbitrary constant. From the definition of ψ_0',

$$\psi_0' = u_0(y) = a_1 y + a_4 y^4 + \ldots,$$

where $\qquad a_1 = \tfrac{1}{2}\alpha, \qquad a_4 = -\tfrac{1}{2}\alpha^2/4!, \quad \ldots$

and $\alpha = 1{\cdot}32824$. Goldstein defines $\alpha_{3r+1} = 3^{3r+2} a_{3r+1}$, for $r = 0, 1, 2, \ldots$. Then

$$a_1 = \alpha_1/9 = a/9Pr,$$

since $a = \alpha_1 Pr$ from (11). Therefore, for sufficiently small y,

$$\Theta_a \sim -Pr^{-1}\{G_1 - \tfrac{1}{2}Pr\, \tfrac{2}{3} a_1^{1/2} y^{3/2}\}^2.$$

If $G_1 = 0$,

$$\Theta_a \sim -\frac{Pr}{9} a_1 y^3 = -\frac{a}{3}\left(\frac{y}{3}\right)^3.$$

This is exactly the form required for the matching. The required solution of Θ_a is then

$$\Theta_a = -\tfrac{1}{4}Pr\left[\int \psi_0'^{1/2}\, dy\right]^2. \tag{34}$$

The solution for Θ_b can be shown to be

$$\Theta_b = -\tfrac{1}{2}Pr\, A\psi_0'^{1/2} \int \psi_0'^{1/2}\, dy + G_2\left[\int \psi_0'^{1/2}\, dy\right]^{2/3},$$

where G_2 is an arbitrary constant. Setting $G_2 = 0$, the behaviour of Θ_b for small y is as required, namely,

$$\Theta_b \sim -a\delta_0(y/3)^2.$$

The required solution for Θ_b is then

$$\Theta_b = -\tfrac{3}{2}Pr\delta_0\psi_0'^{1/2}\int \psi_0'^{1/2}\,dy, \tag{35}$$

since $A = 3\delta_0$.

The required solution for Θ_c is

$$\Theta_c = -\tfrac{9}{4}Pr\delta_0^2\left[\tfrac{1}{2}\psi_0''\psi_0'^{-1/2}\int \psi_0'^{1/2}\,dy + \psi_0'\right]. \tag{36}$$

For sufficiently small y it becomes

$$\Theta_c \sim -a\delta_0(y/3)$$

as required for perfect matching with (30).

The solution for Θ_0 for all y is lengthy but its behaviour at small y can be checked to be

$$\Theta_0 \sim y^{-2}$$

as required for the matching.

Finally, the constant N must be

$$N = 9A_0\exp(-\tfrac{1}{3}a\delta_0^3).$$

Terms of higher order than Θ_0 were not calculated.

Thus a solution of θ for large y has been obtained and matches the solution for small y in the following sense. For a given sufficiently large η, ξ is made so small that only terms independent of ξ are important in the asymptotic solution stemming from the region of small y but large η. The solution for large y matches this form as y gets sufficiently small.

The series (31) does not permit matching with the series solution for small y to terms involving ξ^2 and higher as appeared in (27) for large η. Attempts at modifying the formal expansion (31) to obtain a formal matching to higher order powers of ξ have so far been unsuccessful. This is in sharp contradiction with Goldstein's velocity solution where perfect matching is indicated.

It is rather fortunate, as shown by the computed results, that the physical problem of present interest is in the region of very small ξ and a value of η such that the solution for small y applies. Therefore, the matching procedure, unsatisfactory as just described, recedes in importance except for mathematical interest.

8. LOCATION OF FIRST LOCAL TEMPERATURE MAXIMUM

For a pure mixing interaction between hot and cold flow the temperature distribution in the wake follows a uniformization process. No local maxima can appear since there are no heat sources in the flow field. With heat release by chemical reaction this is no longer true. The distributed chemical reaction raises the temperature throughout the field and with the assumed rate law given in §2 a local maximum in reaction rate occurs in the hot stream thus causing a local temperature maximum to appear there also. This local maximum then rapidly propagates upward into the cool combustible.

With the temperature distribution given by $\theta(\xi, \eta)$, the locus of temperature maxima, $\xi(\eta)$, is given by $\theta_\eta = 0$.

Prior to formation of the initial local maximum, $\theta_{\eta\eta} < 0$ for $\xi < \xi_i$, where ξ_i is the location of the initial maximum. After the maximum forms there must be a value of η at which an inflection occurs, i.e. $\theta_{\eta\eta} = 0$. Above this point $\theta_{\eta\eta} < 0$, and below this point $\theta_{\eta\eta} > 0$. As $\xi - \xi_i \to +0$ the inflection point coalesces with the point where $\theta_\eta = 0$. Therefore, the initial maximum is given by the conditions

$$\theta_\eta = \theta_{\eta\eta} = 0 \tag{37}$$

at $\xi = \xi_i$, $\eta = \eta_i$.

The numerical results indicate that it is sufficiently accurate to use the first two terms of $\theta(\xi, \eta)$ for calculation of ξ_i. From (37)

$$\theta'_0/\theta'_2 = \theta''_0/\theta''_2 \qquad \text{at } \eta = \eta_i,$$

and

$$\xi_i = (-\theta'_0/\theta^1_2)^{1/2}_{\eta=\eta_i}. \tag{38}$$

From (19), (22) and (23)

$$\theta'_0(\eta, \Lambda) = \frac{1-\Lambda}{1-\Lambda^*}\theta'_0(\eta, \Lambda^*), \quad \theta'_2(\eta, \theta_a, B) = \frac{B}{B^*}\exp(\theta^*_a - \theta_a)\theta'_2(\eta, \theta^*_a, B^*),$$

where it has been assumed that the initial maximum occurs where $\theta_0 \doteqdot 1$.

Putting these results into (38) one gets ξ_i in terms of a known ξ^*_i. With the restrictions that $Pr = Pr^*$, $Sc = Sc^*$, this gives

$$\xi_i = \left[\frac{1-\Lambda}{1-\Lambda^*}\,\frac{B^*}{B}\exp(\delta_a - \delta^*_a\right]^{1/2}\xi^*_i. \tag{39}$$

This simple formula (39) gives explicitly the dependence of the critical distance ξ_i on all the important dimensionless quantities, namely, the temperature ratio of the two streams Λ, the activation energy $\theta_a = A/RT_{II}$, and most interesting of all, the parameter $B = 4l\Delta H_{12}/u_I \tau c_p T_{II}$ which combines the fluid mechanical properties l, u_I with the chemical kinetic properties $\Delta H_{12}/c_p T_{II}$ and τ.

The effect of each individual physical quantity can be obtained easily from this formula. A detailed discussion and the implication concerning the appearance of a flame has been given in a previous paper (Cheng & Kovitz 1956).

The following important conclusions may be mentioned.

(i) The presence of a viscous wake shortens, by an order of magnitude, the distance ξ_i where the temperature maximum occurs, as compared with the result without the initial wake (Marble & Adamson 1954).

(ii) The apparent fluid-mechanical variable in determining the distance ξ_i is the shear $\partial u/\partial y$ at the trailing edge of the plate.

(iii) The variation of ξ_i due to thermo-chemical–kinetic properties is overwhelming compared with the variation produced by the fluid-mechanical properties within practical ranges.

9. NUMERICAL METHOD

Since Goldstein's velocity solution involves a numerical tabulation of the functions f_0, f_3, f_6 it is, of course, necessary to determine θ_0, θ_1, θ_2, ... numerically also. The system of equations for the θ_r being linear, it has been possible to extract the parameters Λ from θ_0 and θ_3, and B from θ_2. That is to say, a tabulation of θ_0 and θ_3 for a given Λ yields θ_0 and θ_3 for all Λ from (19) and (26), while a tabulation of θ_2 for given B yields θ_2 for all B from (22). In § 10, it will be seen that when chemical reaction is absent Λ may be extracted from all the non-vanishing coefficients θ_{3r}, $r = 0, 1, 2, \ldots$. The method of trial and error is preferred to the superposition of solutions to meet the boundary conditions at two different points in the numerical integration for higher accuracy.

The numerical integration of θ_0 and K_0 is straightforward since the initial values at $\eta = \bar{\eta} = 0$ are specified as in § 5. The initial value of θ_3 is also fixed from the result of § 6. These functions were obtained from a single trial and error procedure.

First approximations to θ_2 and θ_2' at $\eta = \bar{\eta} = 0$ can be obtained from the matching technique of § 5, but their precise determination was made from a double trial and error procedure. The single pair of values of $\theta_2(0)$ and $\theta_2'(0)$ which give correct behaviour of θ_2 and $\bar{\theta}_2$ were rather quickly localized using a graphical interpolation method. Details are given by Kovitz (1956).

10. SOLUTION WHEN CHEMICAL REACTION IS ABSENT

The absence of chemical reaction implies $B = C = 0$. Within the approximations used, this uncouples the dependence of θ on K as seen from (28). From the general results in § 3 the inhomogeneous term then becomes, for $s \geqslant 3$,

$$I_s = \sum_{\substack{t=3\\ l=1\\ t+3(l-1)=s}} \{(t-3)f_{3l}'\,\theta_{t-3} - (3l+2)f_{3l}\,\theta_{t-3}'\}. \tag{40}$$

It has been shown that $\theta_1 \equiv 0$. Therefore, $I_4 \equiv 0$. By the same reasoning as was used to show that $\theta_1 \equiv 0$, it can be seen that $\theta_4 \equiv 0$.

With $B = 0$, $I_2 = 0$, as seen from the definition of I_2 in § 3. Therefore, as previously shown for θ_1, $\theta_2 \equiv 0$, which implies $I_5 \equiv 0$, which in turn implies $\theta_5 \equiv 0$.

From equation (40),

$$I_6 = -5f_3\,\theta_0' + 3f_6'\,\theta_3 - 8f_6\,\theta_3'.$$

From Goldstein (1930) $f_3 = f_6 = 0$ at $\eta = 0$. Furthermore, it has been shown that $\theta_3(\eta = 0) = 0$ and that θ_0', θ_3, θ_3' are geometrically antisymmetric about $\eta = \bar{\eta} = 0$. Therefore, I_6 is antisymmetric about $\eta = \bar{\eta} = 0$.

A single trial and error numerical integration yields θ_6' at $\eta = 0$ such that the boundary condition

$$\lim_{\eta \to \infty} \theta_6/\eta^6 = 0$$

is met. θ_6 is shown in figure 6 for $\Lambda = 0{\cdot}286$, $Pr = 1{\cdot}00$.

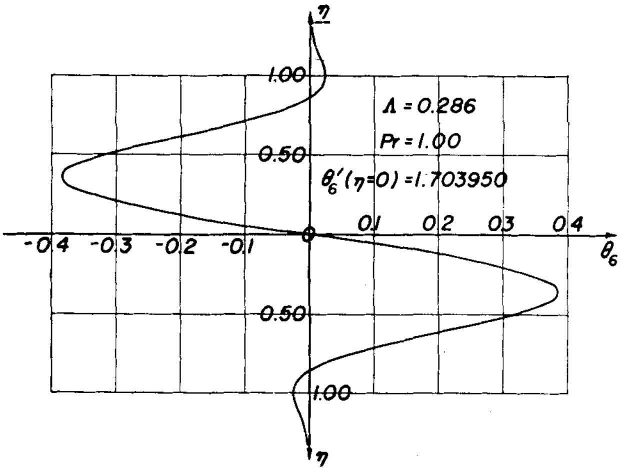

Figure 6. Solution for θ_6.

From the form of I_6 and equations (19) and (26) it is clear that

$$\theta_6(\eta, Pr, \Lambda) = \frac{1-\Lambda}{1-\Lambda^*}\,\theta_6(\eta, Pr, \Lambda^*). \tag{41}$$

From (40) it is seen by induction that I_s depends only on $(s-3)\theta_{s-3}$ and θ'_{s-3} for $s \geqslant 3$. Therefore, only I_{3r} $(r = 1, 2, 3, \ldots)$ is non-vanishing, which implies that only θ_{3r} $(r = 1, 2, 3, \ldots)$ is non-vanishing. The general solution for all Λ in terms of a solution for Λ^* and given Pr is then

$$\theta(\xi, \eta, Pr, \Lambda) = \tfrac{1}{2}(1+\Lambda) + \frac{1-\Lambda}{1-\Lambda^*}\{-\tfrac{1}{2}(1+\Lambda^*) + \theta_0(\eta, Pr, \Lambda^*) + $$
$$+\xi^3\theta_3(\eta, Pr, \Lambda^*) + \xi^6\theta_6(\eta, Pr, \Lambda^*) + \ldots\}. \tag{42}$$

11. CONCLUDING REMARKS

The region of convergence of the series solutions cannot be specified analytically. However, the numerical results do show that with chemical reaction the magnitude of the coefficients $\theta_n(\eta)$ increases rapidly. Since this is essentially a linearized solution only, the region where the effects of chemical reaction are small compared with pure mixing effects can be considered. This is delineated by $0 < \xi < \xi_i$ where ξ_i is again the value of ξ at the appearance of the first local temperature maximum. A detailed computation (Kovitz 1956) using the first four terms of θ, with azomethane as combustible, shows good convergence for $\xi = \xi_i = 0{\cdot}048$. Considering only the dependence of ξ_i on B, given by (40), it can be seen that if the series converges for ξ_i^*, it will converge for ξ_i, since $\xi_i^r\ \theta_r$ is approximately independent of B.

If chemical reaction is absent the coefficients θ_r grow less rapidly. The magnitude of I_r, and so the magnitude of θ_r, depends upon the magnitude of Goldstein's f_{3r}. It may thus be estimated that the convergence without chemical reaction is of the same character as the convergence of Goldstein's solution for u, v.

The authors wish to acknowledge the invaluable aid of Mr R. A. Goerss in setting up and performing the computations. This research is part of the work under OOR DA-36, ORD-2183, sponsored by the Office of Ordinance Research, U.S. Army.

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